AN INTRODUCTION TO EXTREME POINTS AND APPLICATIONS IN ISOMETRIC BANACH SPACE THEORY

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ABSTRACT. This technical paper is the looking at extreme point structure from an isometric view point, within a Banach Space setting. This was a Seminar given to the Analysis group at Goldsmiths College, University of London in May 1998 as part of my early doctoral work.

1. INTRODUCTION AND BASIC DEFINITIONS

In this short paper we look at *extreme points* and important related theorems in finite and infinite dimensions. Extreme points are defined in the setting of any linear space, no topological or metric structure is needed. Minkowski is attributed with first describing such phenomena in his 1911 treatise [8]. We begin with some elementary definitions.

Definition 1. Let L be a linear space over real or complex field $\mathbb{F}$ and let $K \subset L$. We call K *convex* if $\forall\, x, y \in K$,and $\lambda \in (0, 1)$, then $\lambda x + (1 - \lambda)y \in K$.

For example, the unit ball in a normed linear space is convex.

Theorem 1.1. *Let $x \in K$, is an* extreme point *of K if and only if any of the following conditions hold :*

(1) *x is not the inner point of any line segment in K;*
(2) *whenever $x = \lambda y + (1 - \lambda)$ for $\lambda \in (0, 1)$ with $y, z \in K$, then $x = y = z$ (Minkowski,* [8]*);*
(3) *whenever $x = \frac{1}{2}y + \frac{1}{2}z$ for $y, z \in K$,then $x = y = z$.* *(*Mid-point Convexity*);*
(4) *$K \setminus \{x\}$ is convex. (V. Klee* [4]*)*

We denote the set of extreme points of K by ∂K. In the following discussion by a convex set K, we mean that K is also non-empty.

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We introduce some examples of extreme points. In the plane, the vertices of a triangle are extreme points and every point on the circumference of a circle is extremal. In 3- dimensions the vertices of a cube, or a tetrahedron are extreme points. If we take the open triangle in the plane where we exclude its boundary then this set has no extreme points. In $C(X)$ the Banach space of continuous functions on a compact Hausdorff space X, then ± 1 are the only extreme points of the closed unit ball of $C(X)$ where $1(x) = 1$ for all $x \in X$.

Which closed convex sets in $\mathbb{R}^2$ have no extreme points? Well, it is clear that $\emptyset$ and $\mathbb{R}^2$ have no extreme points, similarly lines, unbounded parallel strips, and closed half-planes. In $\mathbb{R}^3$ we would have $\emptyset$ and $\mathbb{R}^3$ with no extreme points, and also lines, planes, and closed half-spaces [11].

Recall that for a real-valued function, f, on a convex set K is called *convex* if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

for all $\lambda \in [0,1]$ and $x, y \in K$. Similarly f is *concave* if $-f$ is convex. If f is both convex and concave, we say f is *affine*.

Letting $A(K)$ denote the Banach space of continuous affine functions on the compact convex set K, where $A(K)$ is endowed with the supremum norm, then the extreme points of the closed unit ball of $A(K)$ are ± 1 where $1(k) = 1$ for all $k \in K$.

To motivate the study of extreme points there are a variety of applications we can cite. For example, Optimisation. The Linear Programming Problem (lpp) to optimise a linear objective function subject to a set of linear constraints has its solution at an extreme point of the convex region bounded by the constraints. Equally the simplex algorithm finds a solution to the lpp by iterating from one extreme point to the next [9]. The Karush-Kuhn-Tucker conditions for optimisation of a non-linear objective function subject to a set of linear constraints also has its solution at an extreme point of the said convex feasible region formed by the constraints [7]. I refer the reader to [9] for many other interesting applications.

Still again in terms of optimisation we have *Bauer's Maximum Principle*:
If f is an upper-semi-continuous convex function on a compact convex set K then there exists an extreme point $x \in K$ such that

$$f(x) = sup_{k \in K} f(k)$$

[1, Thm. I.5.3].

Other motivations will arise as we proceed with the development of the subject.

2. Review of Known Results

We begin this section with a review of well known results. We begin in finite dimensions.

Theorem 2.1. *(Minkowski)*

Let K be a compact convex set in $\mathbb{R}^n$. Each $x \in K$ can be written as a convex combination of extreme points of K, namely

$$x = \sum \lambda_i x_i$$

where $\lambda_i \in (0,1), \sum \lambda_i = 1$ *and* $x_i \in \partial K$.

Notice that if K is a triangle in the plane then such a representation for x is unique but for a square such a representation is not unique. This theorem was originally proved for $\mathbb{R}^3$. Moreover, in finite dimensions, every non-empty closed bounded convex subset has extreme points.

A sharper form of Minkowski's theorem is due to Caratheodory (see [10]).

Theorem 2.2. *(Caratheodory)*

Let K be a compact convex set in $\mathbb{R}^n$. Each $x \in K$ can be written as a convex combination of at most $n+1$ extreme points.

We now move to results concerning infinite dimensional spaces. Let X be a normed linear space, and X_1 its closed unit ball, $X_1 = \{x \in X : \| x \| \leq 1\}$. Let X^* be the dual space of X and X_1^* its closed unit ball, namely $X_1^* = \{x* \in X^* : \| x^* \| \leq 1\}$.

Recall that the Banach -Alaoglu Theorem states that X_1^* is w* compact where the w* topology on X* is given by $x_\alpha^* \longrightarrow x^*$ in the w* topology if and only if $x_\alpha^*(x) \longrightarrow x^*(x)$ for all x $\in X$ (see [2].)

Let E be a locally convex Hausdorff space, in which case every point in E has a local base of neighborhoods of convex sets. For example the weak or norm topologies on E would endow E with a locally convex structure. Notice that for any compact convex subset $K \subseteq E$ with extreme points, then as K contains all of its subsets, and in particular all convex subsets, namely $K = \bar{co}$

$(K) \supseteq \bar{\bar{\text{co}}}$ (∂K). In fact and it turns out that the converse of this is also true, $K \subseteq \bar{\bar{co}}$ (∂K) as stated in the following classical Theorem for extreme points in a locally convex space.

Theorem 2.3. *(Krein-Milman, 1940)* [10]

Let E be a locally convex Hausdorff space, and let K be a convex set. Then $\partial K \neq \emptyset$. Furthermore if K is compact then $K = \bar{\bar{co}}$ (∂K).

Klee reminds us that it is not known if we can replace "K is convex" with "K is *locally convex*" in the above Theorem [6].

To prove the Krein-Milman Theorem, we can make use of the following result, known as Milman's converse.

(Milman's Converse) Let K is a compact convex subset in a locally convex space E. Suppose $Z \subset K$ and $K = \bar{\bar{co}}\, Z$. Then $\partial K \subseteq \bar{\bar{Z}}$.

Proof. Let $x \in \partial K$, then x has a representing probability measure μ on $\bar{\bar{Z}}$, but if x is extremal, x has a unique representing probability measure ε_x and so $\mu = \varepsilon_x$ but then $x \in \bar{\bar{Z}}$ as $x \in \bar{\bar{co}}\, Z$. [10, Prop. 1.2 and 1.4] □

In fact we may restate Krein-Milman as follows : Every $x \in K$ has a unique representing probability measure μ which is supported by $\bar{\bar{\partial K}}$ and its proof now follows easily from Milman's converse.

In terms of $A(K)$ the affine function space, by Krein-Milman's Theorem we can deduce that a continuous affine function on K is completely determined by its values on the extreme boundary of K. A natural question to ask is :

Do we have a Krein-Milman type Theorem for locally compact convex sets? The answer is affirmative and due to Klee [4]. To state this we need the following definitions :

Definition 2. Let K be a in a locally compact convex set in a linear topological space. An *extreme ray* of K is an open half-line ρ such that if $[x, y]$ is a line segment joining x and y (namely $[x, y] = \lambda x + (1 - \lambda)y$, for $0 < \lambda < 1$) then $[x, y] \subset \rho$ whenever $[x, y] \subset K$ and $[x, y]$ meets ρ. We will denote the extreme rays of K by ∂R_K.

Theorem 2.4. *Let K be a closed convex locally compact subset in a linear topological space. Suppose K contains no straight lines, then*

$$K = \bar{\bar{co}}\ (\partial K \cup \partial R_K).$$

We close this section by taking a brief look at the notion of a special type of extreme point, that of an *exposed point*. However unlike extreme points, an exposed point depends on the topology of the underlying space.

Definition 3. A point $x \in K$ is an *exposed point of* K if K is supported at X by a closed hyperplane which intersects K only at x.

Notice that for example for a circle in the plane, every point is an exposed point as there are supporting tangents for every point on the boundary, as well as and each boundary point being extremal. However if we let K be the following set :

$$K = \bar{\bar{co}}\left(\{(x,y) : x^2 + y^2 = 1\} \cup \{(0,3)\}\right),$$

then the points on the circumference where the chords from $(0,3)$ meet the unit circle are extreme points of K but not exposed. We denote the set of exposed points of K by $expK$.

We have the following fundamental theorem due to Straszewicz [11]:

Theorem 2.5. *(Straszewicz, 1935) Every compact convex set in $\mathbb{R}^n$ is the closed convex hull of its exposed points.*

Making use of this theorem we can show that in finite dimensions every extreme point of a closed convex set is the limit of a sequence of exposed points [11].

Klee [5] went on to use this notion of an exposed point with that of an *exposed ray*, denoted by $expR_K$ to prove the following theorem :

Theorem 2.6. *Let K be a closed convex locally compact subset in a normed linear space and suppose K contains no lines. Then*

$$K = \bar{\bar{co}}\left(expK \cup \exp R_K\right).$$

Here an *exposed ray* of K is a closed half-line $\rho \subset K$ such that $\rho = H \bigcap K$ for some closed supporting hyperplane H of K.

In the next seminar we shall take a closer look at exposed points and see that Klee [5] proved a Straszewicz's Theorem for compact convex sets K in a normed linear space, namely that $K = \bar{\bar{co}}\left(expK\right)$.

3. Structure of sets of Extreme Points

In this section we look at sets of extreme points which we can describe topologically within a dimension-free setting (ie infinite dimensions). Firstly we note that an extreme point is an isometric invariant namely:

Theorem 3.1. *Let A and B be real Banach spaces and $T : A \to B$ is a linear isometry. Then T preserves extreme points, namely $T(a)$ is an extreme point of B_1 if and only if a is an extreme point of A_1.*

Thus if the set of extreme points in the unit ball A_1 is closed/open, then the set of extreme points in B_1 is also closed/open if there is a linear isometry between A and B.

A Natural question to ask is :
"Do extreme points always exist"? If K is an open set, it clearly has no extreme points, and we can see that if K is convex and compact then they exist by Krein-Milman.

We can also see that in a Dual Banach space its closed unit ball is w* compact and convex and so it will have extreme points. Can we conjecture that in a Banach space, any closed bounded convex set has extreme points? No, this is false, for example take the closed unit ball of the sequence space of sequences converging to zero, $c_{0,}$ then its closed unit ball is closed and convex but has no extreme points. [3, p.130]. Thus we can see that $c_{0,}$ is not a Dual space. Thus we have a simple test for non-duality of a Banach Space using extreme points :

Test 1. *Let B be a real Banach space and B_1 its closed unit ball. Then B is not a dual space if B_1 has no extreme points.*

We close this paper by showing that under certain circumstances the set of extreme points has a very well-behaved form.

It might seem that the set of extreme points is always a closed set and in fact in $\mathbb{R}^2$ this is **always** the case [11, p.90]. However in higher dimensions this need not be so. For example, even in $3-$ dimensional space we can construct a closed bounded convex set whose extreme points is a closed set.

Take the bi-cone C in the $\mathbb{R}^3$ given by

$$C = \mathrm{co}\left(\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \cup \{(1, 0, 1), (1, 0, -1)\}\right)$$

then the point $(1, 0, 0)$ is not an extreme point and so the set ∂C is open.

We do have the following very satisfactory result

Proposition 3.2. *If X is a metrisable compact convex set in a locally convex Hausdorff space, E, then ∂X is a G_δ set.*

Proof. (See [10]). If X is metrisable then each $x \in X \backslash \{\partial X\}$ can be written as $x = \frac{1}{2}y + \frac{1}{2}z$ for $y, z \in X$, with $d(y, z) > 0$. Let $F_n = \{x \in X : x = \frac{1}{2}y + \frac{1}{2}z$ and $d(y, z) \geq \frac{1}{n}\}$. Now d is continuous and so each F_n is closed. Now $\bigcup_n = 1^\infty = X \backslash \partial X$ and so $X \backslash \bigcup F_n = \partial X = \bigcap (X \ F_n)$ is a G_δ.

□

More generally the structure of the extreme points of a compact convex set may be *pathological*. The well-known "porcupine topology" on K due to Bishop-de Leeuw where the ∂K is a Borel set and even discrete in its relative topology, however the $\sigma-$ algebra generated by ∂K is not equal to K (see [1, Prop. I.4.15]).

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