M-IDEAL PRESERVING MAPS AND BANACH-STONE TYPE THEOREMS

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Abstract. We investigate the properties enjoyed by a surjective linear isomorphism between Banach spaces which preserves M-ideals. We say such maps have Property M. Under property M we show that if T is a linear isomorphism between affine function spaces $A(K)$ and $A(S)$ and every extreme point of K and S are split faces, then ∂K is facially homeomorphic to ∂S. We give examples to show that such a linear isomorphism need not be an isometry and may have arbitrary bound. A key to our result lies in the fact that an M-ideal in $A(K)$ is the annihilator of a closed split face of K. Finally, we begin looking at how to characterise the class of isomorphisms which have property M.

In this paper we begin by considering the notion of an M-ideal within various Banach space and Banach algebra settings. The principle reason for this is to be able to answer the natural question : what type of mapping preserves an M-ideal? (Within a Banach space setting, by a 'map' we always means a surjective linear isomorphism.) Within a ring or algebra structure, the answer is a homomorphism, but if we add a linear structure, what can we say?

We begin with some simple examples showing that such maps need not be isometries and then go on to relate this to a classical problem; namely, we relate M-ideals to a Banach-Stone type theorem for $A(K)$. We conclude this paper by considering candidates for the space of surjective linear isomorphisms satisfying Property M. I warmly thank my supervisor, Professor Cho-Ho Chu,QMW, London, for his enlightened discussions and suggestions for sections 2 and 3.

1. Ideals and M-ideals in Banach spaces and Banach Algebras

We begin with a brief survey. Let E be a complex Banach space, then we call E a *Banach Algebra* if for all $x, y \in E$, we have $x \cdot y \in E$ and E is a normed algebra whose norm satisfies the inequality $\|x \cdot y\| \leq \|x\|\|y\|$. If $E \ni e$ where e is the multiplicative identity, E is called *unital*. A subset $J \subset E$ is called an *ideal* if J is a subspace and $x.y \in J$ for all $x \in E$ and $y \in J$. Further, J is called *maximal* if J is a proper ideal ($J \subsetneq E$) and J is not contained in any larger proper ideal. In a commutative Banach algebra, every proper ideal is contained in a maximal ideal and every maximal ideal is closed. Let X be a compact Hausdorff space and let $C_{\mathbb{C}}(X)$ denote the usual Banach space of continuous complex-valued functions on X together with the supremum norm.

Date: February 2012. Section 3 appeared in my PhD, Goldsmiths College, University of London, 2003. This paper was revised in 2009 and 2012.

A subspace M of $C_{\mathbb{C}}(X)$ a *Function space* if M is uniformly closed in $C_{\mathbb{C}}(X)$, contains the constant functions and separates the points of X. Let $\mathcal{A}$ be a subspace of $C_{\mathbb{C}}(X)$, then we call $\mathcal{A}$ a *Function Algebra* (also called Uniform algebras) if $\mathcal{A}$ is a function space and a subalgebra of $C_{\mathbb{C}}(X)$.

The closed ideals of $C_{\mathbb{C}}(X)$ are precisely the closed algebra ideals and are characterised as

$$\{f : f \in C_{\mathbb{C}}(X) \text{ and } f|_E = 0\}$$

where $E \subset X$ is closed. That is, a closed ideal is the annihilator of a closed subset of X.

The maximal ideals in $C_{\mathbb{C}}(X)$ are sets of the form

$$M_p = \{f \in C_{\mathbb{C}}(X) : f(p) = 0\}$$

for any $p \in X$, and it is well-known that $p \mapsto M_p$ gives a 1-1 correspondence between X and the maximal ideal space $\{\mathcal{M} \in C_{\mathbb{C}}(X) : \mathcal{M} \text{ is a maximal ideal}\}$ with the Gelfand topology.

An Ideal is an algebraic object and a comparable linear structure is an M-ideal.

Let E and F be a real Banach spaces.

Definition 1.1. Let $P : E^* \to E^*$ be a continuous projection, that is, a continuous linear map satisfying $P^2 = P$. We call P an *L-projection* if

$$\|x\| = \|Px\| + \|x - Px\| \quad \forall x \in E^*. \tag{1.1}$$

If P is an L-projection, then $Id-P$ is also an L-projection where Id is the identity operator. Moreover P is a contraction, that is, $\|Px\| \leq \|x\|$.

Definition 1.2. A closed subspace J of E is called an *M-ideal* if the annihilator $J^{\perp} = \{f \in E^* : f(J) = 0\}$ of J is the range of an L-projection on E^*, namely, $J^{\perp} = P(E^*)$ for an L-projection P on E^*.

If P is an L-projection on $A(K)^*$ and x is a state of $A(K)$, namely, $x(1) = 1 = \|x\|$, then since $|(Px)(1)| \leq \|Px\| \leq \|x\|$ and $\|(Id-P)x(1)| \leq \|x - Px\|$, we have

$$\begin{aligned} 1 = x(1) &= (Px)(1) + (Id-P)x(1) = \|x\| \\ &= \|Px\| + \|x - Px\| \geq |(Px)(1)| + |(Id-P)x(1)| \geq 1 \end{aligned}$$

which gives

$$(Px)(1) = \|Px\|. \tag{1.2}$$

In $C_{\mathbb{C}}(X)$ M-ideals are exactly the closed algebra ideals; they are the annihilators of closed subsets of X.

Alfsen and Effros [3] have characterised M-ideals in a Banach space E in terms of the 3-ball property.

Definition 1.3. A linear subspace J of E satisfies the 3-ball property if given 3 open balls B_1, B_2, B_3 in E, for which $B_1 \cap B_2 \cap B_3 \neq \emptyset$, and $B_i \cap J \neq \emptyset$, for $i = 1, 2, 3$, then $B_1 \cap B_2 \cap B_3 \cap J \neq \emptyset$.

The following characterisation is due to Alfsen and Effros [3] :

Theorem 1.1. *Suppose J is a closed subspace of a Banach space E. Then the following are equivalent :*

a) *J is an M-ideal;*

b) *J satisfies the 3-ball property.*

This implies that an isometry between Banach spaces preserves M-ideals. We note that an M-ideal-preserving linear isomorphism need not be an isometry as the following examples will show. Before we are able to discuss the examples, we briefly recall some background.

Throughout this paper K and S are compact convex sets.

Recall that a convex subset F of K is called a *face* of K if $\lambda x + (1-\lambda)y \in F$ for x and y in K and $\lambda \in (0, 1)$, implies that both $x, y \in F$, equivalently, if $K \setminus F$ is convex. If F is a face, a set F' in K is called *complementary* to F if $F \cap F' = \emptyset$ and $K = co(F \cup F')$. Thus each x in K has a decomposition relative to (F, F') namely $x = \lambda y + (1 - \lambda)z$ for some $y \in F$, $z \in F'$ and λ in $[0, 1]$. If F' is a face we call (F, F') a pair of *complementary faces*, and if further λ is *unique*, F is called a *parallel face*. If in addition y and z are unique, then F is called a *split face* and (F, F') is called a pair of *complementary split faces*. The *facial topology on ∂K* is defined by taking

$$\{F \cap \partial K : \ F \text{ is a closed split face of } K\}$$

as the family of all closed sets. The facial topology is weaker than the relative topology on ∂K; it is always T_0, but is T_2 if and only if K is a Bauer simplex [2]. However ∂K is compact in the facial topology. ([1, page 143].)

The following lemma which is straightforward to prove shows that there is a natural characterisation of an M-ideal in $A(K)$, namely as the annihilator of a closed split face of K (c.f. [6]). (We omit the proof for convenience.)

Lemma 1.2. *Let J be a closed subspace of $A(K)$. Then J is an M-ideal if and only if $J = F^{\perp}$ for F a closed split face of K.*

2. Examples of M-ideal-Preserving maps

We now give some examples to investigate the structure of linear isomorphisms which preserve M-ideals.

Example 2.1. Let K be a square in the plane, and S be the pentagon obtained from cutting off a corner of K. Let $T : A(K) \to A(S)$ be the restriction map, then T and T^{-1} preserve M-ideals since the only M-ideals in $A(K)$ and $A(S)$ are the trivial ones, as neither K nor S have any proper closed split face. We note that T can be made to have arbitrary norm by cutting off a suitably sized corner of K.

Example 2.2. Let K be a triangle in the plane, and S be the quadrilateral obtained from cutting off the tip of K. Let $T : A(K) \to A(S)$ be the restriction map, then T^{-1} preserves M-ideals since the only M-ideals in $A(S)$ are the trivial ones. However T does not preserve M-ideals. For example, take $k \in \partial K$, then it's annihilator $\{k\}^{\perp}$ in $A(K)$ is a proper M-ideal but $T(\{k\}^{\perp})$ is not an M-ideal in $A(S)$, being neither the whole of $A(S)$ nor $\{0\}$.

The following example shows that an M-ideal preserving isomorphism between affine functions spaces on Bauer simplexes need not be an isometry.

Example 2.3. Let $X = [1, 2]$ and $Y = [3, 4]$. Let $T : C(X) \to C(Y)$ be defined by

$$Tf(y) = e^y f(y - 2) \qquad (f \in C(X), \quad y \in Y).$$

Then T is a linear isomorphism but not an isometry. In fact

$$\begin{aligned} \|Tf\| \qquad &= \sup\{|e^y f(y-2)| : y \in Y\} \\ &= \sup\{|\ e^{x+2} f(x)| : x \in X\} \\ &\leq e^4 \|f\| \end{aligned}$$

and so $\|T\| \leq e^4$. Also $T^{-1}g(x) = e^{-(x+2)}g(x + 2)$ for all $x \in X$ and $\|T^{-1}\| \leq e^{-3}$. Now T preserves M-ideals since if J is an M-ideal in $C(X)$ with $J = F^{\perp}$ where F is a closed subset of X, then $T(J) = G^{\perp}$ where $G = F + 2$ is a closed subset of Y. Likewise, T^{-1} preserves M-ideals.

Let $\mathcal{A}$ be a function algebra on a compact Hausdorff space X,and $S_{\mathcal{A}}$ be the state space of $\mathcal{A}$. Let $Z_{\mathcal{A}} = co(S_{\mathcal{A}} \cup -iS_{\mathcal{A}})$ be the complex state space of $\mathcal{A}$. The map $\theta : \mathcal{A} \to A(Z_{\mathcal{A}})$ defined by

$$\theta f(z) = re\ z(f) \qquad (f \in \mathcal{A}, z \in Z_{\mathcal{A}})$$

is a real linear isomorphism [4, page 146]. Recall that a function algebra $\mathcal{A}$ is *antisymmetric* if the conditions $f \in \mathcal{A}$ and f is real-valued imply that f is constant [10, page 172]. Note that every extreme point of $Z_{\mathcal{A}}$ is a split face [9].

Our next example is of a non-isometric M-ideal preserving linear isomorphism $T : A(Z_{\mathcal{A}}) \to A(Z_{\mathcal{A}})$.

Example 2.4. Let $X = \overline{\Delta} \times [0,1]$ where Δ is the open unit disk in $\mathbb{C}$. Let $\mathcal{A}$ be the set of functions $f \in C_{\mathbb{C}}(X)$ such that $z \to f(z,t)$ is analytic in Δ, for each $t \in [0,1]$. Then $\mathcal{A}$ is not antisymmetric. (See [10, page 177].) Choose an element $f \in \mathcal{A}$ such that f is real-valued, non-constant, and invertible. Define $T : A(Z_{\mathcal{A}}) \to A(Z_{\mathcal{A}})$ by $Tg = \theta(f\theta^{-1}(g))$. Then T is a linear isomorphism and $T^{-1} = \theta f^{-1}\theta^{-1}$. Let J be an M-ideal in $A(Z_{\mathcal{A}})$ with $J = F^{\perp}$ where F is a closed split face of $Z_{\mathcal{A}}$. For $g \in J$ with $g = \theta h$ and $h \in \mathcal{A}$, we have $Tg(z) = ref(z)z(h) = f(z)rez(h) = f(z)g(z) = 0$ for $z \in F$ and so $Tg \in F^{\perp} = J$ and $TJ \subseteq J$. Also $J \supseteq T^{-1}(J)$. Indeed, for $g \in J$ with $g = \theta h$ and $h \in \mathcal{A}$, we have $T^{-1}g(z) = \theta(f^{-1}\theta^{-1}g(z)) = \theta(f^{-1}(z)h(z)) = re(f^{-1}(z)h(z)) = f^{-1}(z)reh(z) = f^{-1}(z)g(z) = 0$ for $z \in F$. Thus $T^{-1}g \in F^{\perp} = J$. So T preserves M-ideals. We note that T is not an isometry if $\|f\| < 1$, say.

3. Property M and a Banach-Stone type Theorem for $A(K)$

It is known that an isometry between $A(K)$ and $A(S)$ always induces a homeomorphism between ∂K and ∂S, for example see [8]. With this in mind, we ask what happens if we replace an isometry by a linear isomorphism? Chu and Cohen [7] have proved that if ∂K and ∂S are closed and every every extreme point of K and S is a split face, then a bound-2 isomorphism from $A(K)$ to $A(S)$ yields ∂K and ∂S homeomorphic. Jarosz [11] has proved an analogous result for function algebras, namely whenever there is a bound-2 complex linear isomorphism from $\mathcal{A}$ onto $\mathcal{B}$, then their Choquet boundaries, $ch(\mathcal{A})$ and $ch(\mathcal{B})$ are homeomorphic. For a function algebra $\mathcal{A}, \overline{re\mathcal{A}}$ is linearly isometric to $A(K)$, where $re\mathcal{A}$ denotes the real part of $\mathcal{A}$ and K is the state space of $\mathcal{A}$, and in this setting, $ch(\mathcal{A})$ is homeomorphic to ∂K and every extreme point is a split face.

Definition 3.1. We say a linear isomorphism T from $A(K)$ onto $A(S)$ satisfies *Property M* if both T and T^{-1} preserve M-ideals.

We note that such a linear isomorphism need not be an isometry and in fact, it may have arbitrary bound, as examples above have shown. The class of surjective linear isomorphisms on a Function space satisfying Property M is thus larger than the class of isometries on it.

Our result in this section proves that if T is a linear isomorphism from $A(K)$ onto $A(S)$ which satisfies Property M, and if every extreme point of K and S are split faces, then ∂K is facially homeomorphic to ∂S. A key to the result lies in the well-known fact that an M-ideal in $A(K)$ is the annihilator of a closed split face of K.

Our first two lemmas gives simple identifications of the maximal M-ideals in $A(K)$.

Lemma 3.1. *Let $k \in \partial K$. If $\{k\}$ is a split face of K, then*

$$J_k = \{k\}^\perp = \{f \in A(K) : f(k) = 0\}$$

is a maximal M-ideal.

Proof. Let J be a proper M-ideal in $A(K)$ and $J \supseteq J_k$. Then $J^\perp$ is a split face of K and

$$J^\perp \subseteq J_k^\perp = \{k\}^{\perp\perp} = \{k\}$$

and so $J^\perp = \{k\}$. Hence $J = J^{\perp\perp} = \{k\}^\perp$. So J_k is maximal. □

Lemma 3.2. *Suppose that every extreme point of K is a split face. Then every maximal M-ideal in $A(K)$ is of the form J_k.*

Proof. By Lemma 1.2, every M-ideal in $A(K)$ is of the form J_F, where F is a closed split face of K and

$$J_F = \{f \in A(K) : f|_F \equiv 0\}.$$

Suppose J_F is a maximal M-ideal in $A(K)$. Let $k \in \partial F$. Then $\{k\}^\perp \supseteq J_F$ and by the maximality of J_F, we have $\{k\}^\perp = J_F$. □

Let $Max(A(K))$ be the set of all maximal M-ideals in $A(K)$. We topologise $Max(A(K))$ with the hull-kernel topology as follows.

Let $J \subseteq Max(A(K))$. We define the *hull* $hull(J)$ of J to be :

$$hull(J) = \{M \in Max(A(K)) : M \supseteq J\}.$$

If $\mathcal{S} \subseteq Max(A(K))$, then the *kernel* $\ker(\mathcal{S})$ of $\mathcal{S}$ is defined to be the largest M-ideal contained in $\cap\{J : J \in \mathcal{S}\}$.

It can be shown that for $\mathcal{S} \in Max(A(K))$, $hull(\ker(\mathcal{S}))$ defines the closure operation of a topology on $Max(A(K))$, called the *hull-kernel topology* (c.f. [4, page 225]).

We begin with the following lemma.

Lemma 3.3. *Let K and S be compact convex sets and suppose every extreme point of K and S is a split face. Let $T : A(K) \to A(S)$ be a surjective linear isomorphism which satisfies Property M. Then the map $\Phi : Max(A(K)) \to Max(A(S))$ defined by*

$$\Phi(J) = T(J), \quad \textit{for} \quad J \in Max(A(K)),$$

is a homeomorphism.

Proof. Since T and T^{-1} preserve M-ideals, then T and T^{-1} also preserve maximal M-ideals. Also T is a bijection. Hence Φ is well-defined.

If $\mathcal{S} \subseteq Max(A(K))$ then $T(ker(\mathcal{S}))$ is an M-ideal and therefore,

$$\begin{aligned} T(\mathit{ker}(\mathcal{S})) &\subseteq T(\cap\{J : J \in \mathcal{S}\}) \\ &= \cap\{TJ : J \in \mathcal{S}\} \\ &= \cap\{\Phi J : J \in \mathcal{S}\} \\ &= \cap\{J : \Phi^{-1}J \in \mathcal{S}\}, \end{aligned}$$

which implies that $T(\mathit{ker}\mathcal{S}) \subset \mathit{ker}(\Phi\mathcal{S})$. Applying the same argument to T^{-1} and Φ^{-1} we have

$$T^{-1}(\mathit{ker}(\Phi(\mathcal{S}))) \subseteq \mathit{ker}(\Phi^{-1}(\Phi(\mathcal{S}))) = \mathit{ker}(\mathcal{S}),$$

and so

$\mathit{ker}(\Phi(\mathcal{S})) \subset T(\mathit{ker}(\mathcal{S}))$ and so we have equality. Thus, if $\overline{\mathcal{S}}$ is the closure of $\mathcal{S}$ in the hull-kernel topology, then we have

$$\begin{aligned} \Phi(\overline{\mathcal{S}}) &= \Phi(\{J : J \supseteq \mathit{ker}(\mathcal{S})\}) \\ &= \{TJ : J \supseteq \mathit{ker}(\mathcal{S})\} \\ &= \{TJ : T(J) \supseteq \mathit{ker}(\Phi(\mathcal{S}))\} \\ &= \mathit{hull}(\mathit{ker}(\Phi(\mathcal{S}))) = \overline{\Phi(\mathcal{S})}. \end{aligned}$$

Thus Φ is a closed map. A similar argument shows that Φ^{-1} is closed, and so Φ is a homeomorphism. □

Theorem 3.4. *Let K and S be compact convex sets such that each extreme point in K and S are split faces. If there exists linear isomorphism $T : A(K) \to A(S)$ which satisfies Property M, then ∂K is homeomorphic to ∂S in the facial topology.*

Proof. Let $\tau : \partial K \to Max(A(K))$ be defined by $\tau(k) = J_k$ for each $k \in \partial K$. Then τ is a bijection by Lemma 3.2. Let $\mathcal{S}$ be a closed subset of $Max(A(K))$. Then $\mathcal{S} = hull(ker(\mathcal{S}))$. Let F be the smallest closed split face of K containing $\{k \in \partial K : \{k\}^{\perp} \in \mathcal{S}\} = \{k \in \partial K : k \in \tau^{-1}(\mathcal{S})\}$. Then $F^{\perp}$ is the largest M-ideal contained in $\bigcap\{\{k\}^{\perp} : \{k\}^{\perp} \in \mathcal{S}\}$, that is, $ker(\mathcal{S}) = F^{\perp}$. We show that $\tau^{-1}(\mathcal{S}) = \partial F$ which is therefore facially closed. By definition of F, we have $\tau^{-1}(\mathcal{S}) \subseteq \partial F$. Conversely, if $k \in \partial F$ then $\{k\} \subseteq F$ and so $\tau(k) = \{k\}^{\perp} \supseteq F^{\perp}$ hence $\tau(k) \in hull(ker((\mathcal{S})) = \mathcal{S}$. Thus $\tau^{-1}(\mathcal{S}) \supseteq \partial F$. This proves that τ is continuous.

To show that τ is an open map, let U be a facially open set in ∂K, then $U = \partial K \backslash \partial F$ where F is a closed split face of K. We will show that $Max(A(K)) \backslash \tau(U)$ is closed, namely that $Max(A(K)) \backslash \tau(U)$ contains its hull-kernel. Now $\tau(U) = \{\{k\}^{\perp} : k \in U\}$ and so if $\{k\}^{\perp} \in Max(A(K)) \backslash \tau(U)$ then $k \notin U = \partial K \backslash \partial F$ and so $k \in \partial F$. That is,

$$\{\ k : \{k\}^{\perp} \in Max(A(K)) \backslash \tau(U)\ \} \subseteq F.$$

By definition of $ker(Max(A(K)) \backslash \tau(U))$, its annihilator is the smallest closed split face containing $\{k : \{k\}^{\perp} \in Max(A(K)) \backslash \tau(U)\}$, and so

$$(ker(Max(A(K)) \backslash \tau(U))^{\perp} \subseteq F.$$

Next, suppose

$$\{k\}^{\perp} \in\ hull(ker(Max(A(K)) \backslash \tau(U)), \quad \text{for some} \quad k \in \partial K.$$

Then $\{k\}^{\perp}$ contains $ker(Max(A(K)) \backslash \tau(U))$, and so,

$$\{k\} \subseteq (ker((Max(A(K)) \backslash \tau(U)))^{\perp} \subseteq F.$$

Hence $k \notin U$ and $\{k\}^{\perp} \in Max(A(K)) \backslash \tau(U)$. Therefore $Max(A(K)) \backslash \tau(U)$ is closed and τ is a homeomorphism.

Finally $\rho : Max(A(S)) \to \partial S$ defined by $\rho(J_s) = s$ for every maximal M-ideal in $Max(A(S))$ is a homeomorphism, as before.

Thus $\sigma = \rho \circ \Phi \circ \tau$ is a homeomorphism from ∂K onto ∂S, where ∂K and ∂S have the facial topology. □

Corollary 3.5. *Under the conditions of the above Theorem, the centre $Z(A(K))$ of $A(K)$ is linearly isometric to $Z(A(S))$.*

Proof. The homeomorphism $\partial K \to \partial S$ induces a linear isometry between $Z(A(K)) = C(\partial K)$ and $Z(A(S)) = C(\partial S)$.

□

Behrends [5, Ch7] gives three proofs of the classical Banach-Stone theorem using isometric invariants, one of which uses M-ideals. (It was Behrend's book which was the starting point for this paper.) We can now give a re-formulation of this classical theorem, using the results in this paper.

Corollary 3.6. *Let X and Y be compact Hausdorff spaces. If there exists a linear isomorphism T from $C(X)$ onto $C(Y)$ such that T satisfies Property M, then X and Y are homeomorphic.*

Proof. This follows from the fact that $C(X)$ identifies with $A(K)$, where K is a Bauer simplex and X is homeomorphic to ∂K. □

Remark 3.1. The linear isomorphism T in the above theorem need not be an isometry (c.f. Example 2.3).

A natural question to is to ask whether there is any relationship between a homeomorphism in the facial topologies on ∂K and ∂S and a homeomorphism in the relative topologies. The answer is negative as the following simple examples show.

If ∂K and ∂S are relatively homeomorphic, it does not necessarily follow that they are facially homeomorphic.

Example 3.1. Let K be the unit square in $\mathbb{R}^2$ and S be the tetrahedron in $\mathbb{R}^3$. Then ∂K and ∂S are relatively homeomorphic. However, the facial topology on ∂K has the indiscrete topology whilst the facial topology on S is not indiscrete.

If ∂K and ∂S are facially homeomorphic, it does not necessarily follow that they are relatively homeomorphic.

Example 3.2. Let K be a semi-circle in the plane, and S be the bi-cone in $\mathbb{R}^3$ given by $S = \mathrm{co}(\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \cup \{(1,0,1),(1,0,-1)\})$. It is clear that there is a bijection between ∂K and ∂S and as their facial topologies are indiscrete, they are facially homeomorphic. However they are not relatively homeomorphic as ∂K is closed but ∂S is not.

4. Property M : an M-ideal preserving map

This section contains numerous open questions and conjectures. Recall :

Definition 4.1. We say that a surjective linear isomorphism T between Function spaces E and F satisfies *Property* M if T is M-ideal preserving, that is, both T and T^{-1} preserve M-ideals in E and F.

As we have seen above in section 3, such a linear isomorphism need not be an isometry and may have arbitrary bound. The class of surjective linear isomorphisms on a Function space satisfying Property M is thus strictly larger than the class of isometries on it.

The question of interest is: can we characterise Property M in a natural way?

We consider two possible natural candidates, *the Bounded Extension Property* and the *Best Approximant Property.*

4.1. **Bounded Extension Property.**

Let X be compact T_2 space and $Y \subset X$ be closed. The pair (H, L) of subspaces of $C(X)$ and $C(Y)$ respectively, has the *bounded extension property* (B.E.P.) if there is a constant C such that for every $\varepsilon > 0$ and every open set $O \supseteq Y$ and every $f \in H$ there is a $g \in L$ such that

$$\begin{aligned} \|g\| &\leq C\|f\|; \\ g|_Y &= f; \\ |g(x)| &\leq \varepsilon\|f\| \quad \forall x \in X \backslash O. \end{aligned}$$

See for example, [12, Ch III.D]. The following lemma follows.

Lemma 4.1. *If* (H, L) *has the B.E.P. then*

$$H_0 = \{f \in H : f|_X = 0\}$$

is an M-ideal in H.

The following seems natural.

Proposition 4.2. *If* $T : C(X) \mapsto C(Y)$ *is a linear isomorphism and* (H, L) *has the B.E.P. then* T *and* T^{-1} *preserves M-ideals in* H *and* L.

An analogous property in $A(K)$ would be as follows. Noting that, as there is a well-known B.E.P. for $A(K)$ (see, for example, [1, II.5]), we shall call this notion, the *M-Bounded Extension Property* (M.B.E.P.).

Let K be compact convex set and $F \subset K$ be a closed split face. The pair (H, L) of subspaces of $A(K)$ and $A(F)$ respectively, has the *M-bounded extension property* (M.B.E.P.) if there is a constant C such that for every $\varepsilon > 0$ and every open set $O \supseteq F$ and every $f \in H$ there is a $g \in L$ such that

$$\begin{array}{rcl} \|g\| & \leq & C\|f\|; \\ g|_F & = & f; \\ |g(k)| & \leq & \varepsilon\|f\| \quad \forall k \in K \backslash O. \end{array}$$

The following lemma follows.

Lemma 4.3. *If (H, L) has the M.B.E.P. then*

$$H_0 = \{f \in H : f|_F = 0\}$$

is an M-ideal in H.

Question 1 : If F has an extreme point k, then does the M.B.E.P. imply that k is a weak peak point for $A(K)$? We conjecture affirmatively. (If k is a w.p.p., this is a weaker condition than k is a split face.)

[Recall $k \in \partial K$ is called a *weak peak point* for $A(K)$ if whenever $1 > \varepsilon > 0$ and U is an open subset of K then there is a function $h \in A(K)$ with $\|h\| \leq 1,\ h(k) > 1 - \varepsilon$ and $|h(x)| < \varepsilon$ for all $x \in \partial K \backslash U$.]

Question 2: can we link this to a property that says a mapping T is M-ideal preserving? We conjecture that this is likely.

4.2. **Best Approximant.** Let E be a Banach space and let J be an M-ideal in E. For each $\varphi \in E^*$ if there is one and only one $\varphi_0 \in J^\perp$ such that

$$\|\varphi - \varphi_0\| = inf\|\varphi - \varphi^\perp\|$$

where the infimum is taken over all $\varphi^\perp \in J^\perp$, then we say φ_0 is the *best approximant* to φ.

Lemma 4.4. *Let J be an M-ideal in $A(K)$ and so $J^\perp$ is an L-summand in $A(K)^*$ with P the associated L-projection on $A(K)^*$. Then each $\varphi \in A(K)^*$ has one and only one* best approximant*, namely $\varphi_0 = P(\varphi)$.*

Question 3: can we link this to a property that says a mapping T is M-ideal preserving? We conjecture affirmatively. One idea would be as follows. Suppose $J \subset A(K)$ and $J' \subset A(S)$ are M-ideals and T is a linear isomorphism between $A(K)$ and $A(S)$. Then T^* will map each element $\varphi \in A(S)^*$ onto a unique point $\varphi_0 \in J'^{\perp}$ which is of minimal distance from the given φ.

February 2012

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