The Unique Decompostion Property and the Banach-Stone Theorem

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The Classical Banach-Stone Theorem :

Theorem. *Let X and Y be compact Hausdorff spaces. Then $C(X) \cong C(Y)$ if and only if $X \simeq Y$.*

Definition. *A compact convex set K in a locally convex space, E, is a* **Choquet simplex** *whenever for all $x \in E$ and $\alpha > 0$ the set $K \cap (x + \alpha K)$ is either empty or of the form $y + \beta K$ for some $y \in E$ and $\beta \geq 0$. If, in addition, ∂K is closed, K is a* **Bauer simplex**.

Let ∂K denote the extreme points of K.

Let $f : \partial K \to R$ be continuous.
K Bauer simplex $\Rightarrow$ f has a unique affine cts extension to K, ie $C(\partial K) \cong A(K)$.

In the context of affine geometry :

Theorem. *Let K and S be Bauer simplexes. Then $A(K) \cong A(S)$ if and only if K is affinely homeomorphic to S.*

Known Results for Thm

- Lazaar (1968) proved for K, S Choquet simplexes

- Ellis and So (1987) proved for K, S with the property that every pair of closed complementary faces is split.

New results

Let K and S be compact convex sets.

Theorem. *If S is* **Skew-symmetric**, *then every isometry $T : A(K) \to A(S)$ induces an affine homeomorphism between K and S.*

- We also prove the converse.

- We also prove that every isometry is a weighted composition operator modulo a skew isometry.

Definition. *A convex subset F of K is called a* **face** *if for any $x \in F$ with $x = \lambda y + (1-\lambda)z$ for $\lambda \in (0,1)$ and $y, z \in K$ then $y, z \in F$.*

Definition. *If λ is unique, for each $x \in K \setminus (F_1 \cup F_2)$, then (F_1, F_2) are called* **parallel faces** *of K. If in addition y and z are unique then (F_1, F_2) are called* **split faces** *of K.*

Note : we always embed K in $A(K)^*$, and so closed unit ball of $A(K)^*$ is
$B_{A(K)^*} = (K \cup -K)$,

The following results are simple but key to this paper:

Lemma. *Let K and S be compact convex sets and let $T\colon A(K) \to A(S)$ be a surjective linear isometry. Then $T1(s) = \pm 1$ for all $s \in \partial S$.*

Note: $T^* : A(S)^* \to A(K)^*$ is a linear isometry, hence $T^* : \partial S \cup \partial(-S) \to \partial K \cup \partial(-K)$. Thus for each $s \in \partial S$ if $T^*s \in K$, then $T^*s(1) = 1$. Equally, if $T^*s \in -K$, $T^*s(-1) = 1$, and so $T1(s) = T^*s(1) = \pm 1$.

Note: If $T : A(K) \to A(S)$ a surjective linear isometry then $S_1 = \{s \in S : T1(s) = 1\}$ and $S_2 = \{s \in S : T1(s) = -1\}$ is a pair of **Parallel faces of S associated with $T1$.**

Definition. *We call T a* **composition operator** *whenever there is a continuous affine mapping $\sigma\colon S \to K$ such that $Tf = f \circ \sigma$.*

If σ is an affine homeomorphism then T is a surjective linear isometry with $T1 = 1$. The converse holds :

Lemma. *Let $T\colon A(K) \to A(S)$ be a linear isometry with $T1 = 1$. Then T is a composition operator $f \mapsto f \circ \sigma$ where $\sigma\colon S \to K$ is an affine homeomorphism.*

Definition. *Let (S_1, S_2) be a pair of closed parallel faces of S. The* **skew associate S' of S with respect to** (S_1, S_2)*is*

$$S' = (S_1 \cup -S_2).$$

Definition. *For each* $f \in A(S')$, **The natural skew isometry** $T'\colon A(S') \to A(S)$ *is defined to be*

$$T'f(\lambda s_1 + (1-\lambda)s_2) = \lambda f(s_1) + (\lambda - 1)f(-s_2),$$

for all $s_1 \in S_1$, $s_2 \in S_2$ *and* $0 \leq \lambda \leq 1$.

Let S' be a skew associate of S and let $T'\colon A(S') \to A(S)$ be the natural skew isometry. Then every affine homeomorphism $\sigma\colon S' \to K$ induces a surjective linear isometry $T\colon A(K) \to A(S)$ by defining $Tf = T'(f \circ \sigma)$ for all $f \in A(K)$. Conversely,

Theorem. *Every surjective linear isometry* $T\colon A(K)$ $A(S)$ *is of the form*

$$Tf = T'(f \circ \sigma), \qquad (\forall f \in A(K))$$

where $\sigma : S' \to K$ *is an affine homeomorphism, and* $T' : A(S') \to A(S)$ *is the natural skew isometry. The skew associate* S' *of* S *is with respect to the pair of closed parallel faces* (S_1, S_2) *associated with* $T1$.

Definition. S *are* **skew symmetric** *whenever every skew associate of* S *is affinely homeomorphic to* S*.*

Also, every linear isometry $T : A(K) \to A(S)$ induces an affine homeomorphism between K and a skew associate S' of S. Thus if S is skew-symmetric, $S' \simeq S$ then $K \simeq S$. This proves our first Banach–Stone type Theorem, Theorem

This extends the results of Lazar and Ellis and So. Notice : if S is not skew-symmetric, then an isometry T need not induce and affine homeomorphism between K and S, as the following example due to J.T. Chan shows:

Let K be a 3-dimensional triangular prism in R^4 and S be the octahedron in R^4. If the extreme points of K are:

a= b = c =

d = e = f =

then the extreme points of S are $\{a, b, c, -d, -e, -f\}$

Then $K = co(F_1 \cup F_2)$ and $S = co(F_1 \cup -F_2)$. Then S is a skew-associate of K, $A(K) \cong A(S)$ and yet K is not affinely homeomorphic to S.

We also have the converse :

Theorem. *If* $T : A(K) \to A(S)$ *induces and affine homeomorphism between* K *and* S*, then* S *is skew-symmetric*

Our second Banach–Stone type theorem.

Theorem. *Let S be compact convex set. Then the following are equivalent:*

$a)$ *every closed parallel face of S is split;*

$b)$ *every linear isometry T from any $A(K)$ onto $A(S)$ is a weighted composition operator.*

Corollary. *Let S be a compact convex set. Suppose that every closed parallel face of S is split. Then S is skew symmetric.*

Let K be a general quadrilateral in R^2 with no geometrically parallel face. Then K property does not satisfy the Ellis and So condition as it has complementary faces which are not split. It has trivial parallel faces $(K, \emptyset)$ which are split, hence by , K is skew-symmetric.

Let K be a hexagon in R^2. Then K satisfies the condition of Ellis and So because it has no proper complementary faces.

Let K be the convex set formed by cutting a hexagonal cylinder by two horizontal non-parallel planes. The two hexagonal faces are complementary but not parallel or split, and hence K does not satisfy the Ellis and So condition. Indeed, the only parallel faces are $\emptyset$ and K which are split, and thus every linear isometry onto $A(K)$ is a weighted composition operator.

Let K be a square in R^2. Then, as above, K does not satisfy the condition of Ellis and So. We also see that K has parallel faces that are not split. An elementary analysis reveals that $A(K)$ is linearly isometric to R^3 with ℓ^1 norm; so that, $B_{A(K)}$ is an octahedron. The functions $h \in B_{A(K)}$ with $h(x) = \pm 1$ for all $x \in \partial K$ are, in this case, just the extreme points of $B_{A(K)}$. But K is skew symmetric from the results of this paper.